ED Serie 1 Sebastian Keller

Vektoridentitäten 1.1

- $a \cdot (b \wedge c)_i = a_i \cdot (b \wedge c)_i = a_i \, \varepsilon_{ijk} \cdot b_j c_k$

$= - \varepsilon_{ikj} \, a_i b_j c_k = b_j \, \varepsilon_{jki} \, c_k a_i = b \cdot (c \wedge a)$

analog für $c \cdot (a \wedge b)$

- $(a \wedge (b \wedge c))_i = \varepsilon_{ijk} \, a_j \, (b \wedge c)_k = \varepsilon_{ijk} \, a_j \, \varepsilon_{klm} \, b_l c_m$

$= a_j \, \varepsilon_{ijk} \varepsilon_{klm} \, b_l c_m = a_j \, \varepsilon_{kij} \, \varepsilon_{klm} \, b_l c_m =$

$a_j \delta_{il} \delta_{jm} \, b_l c_m - a_j \, \delta_{im} \delta_{jl} \, b_l c_m = (a_j \cdot c_j) \cdot b_i - (a_j \cdot b_j) \, c_i$

$= (a \cdot c) b - (a \cdot b) c$

- $(a \wedge b) \cdot (c \wedge d) = \varepsilon_{ijk} \, a_j b_k \cdot \varepsilon_{imn} \, c_m d_n$

$= a_j b_k \left(\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \right) c_m d_n = a_j b_k c_j d_k - a_j b_k c_k d_j$

$= (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$

- $\underline{(\mathrm{rot}\,\mathrm{grad}\,\psi)_i} = \varepsilon_{ijk}\,\partial_j\partial_k\psi \qquad \left(\partial_j := \frac{\partial}{\partial x_j}\right)$

$$= \sum_{\substack{\sigma\in S_3\\ \sigma(1)=i}} \varepsilon_{\sigma(1)\sigma(2)\sigma(3)}\,\partial_{\sigma(2)}\partial_{\sigma(3)}\psi$$

$$= \sum_{\substack{\sigma\in S_3\\ \sigma(1)=i\\ \sigma \text{ gerade}}} \varepsilon_{\sigma(1)\sigma(2)\sigma(3)}\,\partial_{\sigma(2)}\partial_{\sigma(3)}\psi + \varepsilon_{\sigma(1)\sigma(3)\sigma(2)}\,\partial_{\sigma(3)}\partial_{\sigma(2)}\psi$$

$$= \sum_{\substack{\sigma\in S_3\\ \sigma(1)=i\\ \sigma \text{ gerade}}} \underbrace{\varepsilon_{\sigma(1)\sigma(2)\sigma(3)}\,\partial_{\sigma(2)}\partial_{\sigma(3)}\psi - \varepsilon_{\sigma(1)\sigma(2)\sigma(3)}\,\partial_{\sigma(3)}\partial_{\sigma(2)}\psi}_{=0}$$

$$= 0$$

- $\underline{\mathrm{div}\,\mathrm{rot}\,A_i} = \partial_i\,\varepsilon_{ijk}\,\partial_j A_k = \varepsilon_{ijk}\,\partial_i\partial_j A_k = \varepsilon_{kij}\,\partial_i\partial_j A_k = 0$

(siehe oben)

- $\underline{\mathrm{rot}(\mathrm{rot}\,A)_i} = \varepsilon_{ijk}\,\partial_j(\mathrm{rot}\,A_k) = \varepsilon_{ijk}\,\partial_j\,\varepsilon_{kmn}\,\partial_m A_n = \varepsilon_{kij}\,\varepsilon_{kmn}\,\partial_j\partial_m A_n$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\,\partial_j\partial_m A_n = \delta_{im}\,\partial_j\partial_m A_j - \delta_{in}\,\partial_j^2 A_n$$

$$= \partial_i\partial_j A_j - \partial_j^2 A_i = \mathrm{grad}(\mathrm{div}\vec{A}) - \Delta\vec{A}$$

- $\underline{\mathrm{div}(\psi\vec{A})} = \partial_i(\psi A_i) = \partial_i\psi\cdot A_i + \psi\,\partial_i A_i = \vec{A}\cdot\nabla\psi + \psi\,\mathrm{div}\vec{A}$

- $\underline{\mathrm{rot}(\psi A)_i} = \varepsilon_{ijk}\,\partial_j(\psi A_k) = \varepsilon_{ijk}(\partial_j\psi)A_k + \varepsilon_{ijk}\,\psi\,\partial_j A_k$

$$= \varepsilon_{ijk}\,(\mathrm{grad}\psi)_j A_k + \psi\cdot\varepsilon_{ijk}\,\partial_j A_k$$

$$= \mathrm{grad}\psi \wedge A + \psi\cdot\mathrm{rot}A$$

- $\underline{\mathrm{grad}(A\cdot B)} = (A\cdot\nabla)B + (B\cdot\nabla)A + A\wedge\mathrm{rot}B + B\wedge\mathrm{rot}A = *$

$$(*)_i = \underbrace{A_j\partial_j B_i + B_j\partial_j A_i}_{} + \varepsilon_{ijk}A_j(\mathrm{rot}B)_k + \varepsilon_{ijk}B_j(\mathrm{rot}A)_k$$

$$= \ldots + \varepsilon_{ijk}A_j\,\varepsilon_{kmn}\partial_m B_n + \varepsilon_{ijk}B_j\,\varepsilon_{kmn}\partial_m A_n$$

$$= \ldots + A_j\,\varepsilon_{kij}\varepsilon_{kmn}\partial_m B_n + B_j\,\varepsilon_{kij}\varepsilon_{kmn}\partial_m A_n$$

$$= \ldots + A_j(\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in})\partial_m B_n + B_j(\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in})\partial_m A_n$$

$$= \ldots + A_j\delta_{jn}\partial_i B_n - A_j\delta_{in}\partial_j B_n + B_j\delta_{jn}\partial_i A_n - B_j\delta_{in}\partial_j A_n$$

$$= \ldots + A_j\partial_i B_j - A_j\partial_j B_i + B_j\partial_i A_j - B_j\partial_j A_i$$

$$= A_j\partial_i B_j + B_j\partial_i A_j = \partial_i(A_j B_j) = \mathrm{grad}(A\cdot B)_i$$

- $\underline{\mathrm{div}(A\wedge B)} = \partial_i(\varepsilon_{ijk}A_j B_k) = \varepsilon_{ijk}(\partial_i A_j)B_k + \varepsilon_{ijk}A_j\partial_i B_k$

$$= B_k\,\varepsilon_{kij}\partial_i A_j - A_j\,\varepsilon_{jik}\partial_i B_k = B\,\mathrm{rot}A - A\,\mathrm{rot}B$$

Dirac'sche Delta-Funktion 1.2

a) $\cdot\ \rho(r,\varphi,\vartheta) = \frac{Q}{4\pi R^2}\,\delta(r-R)$

$\Rightarrow$ Ladung $= \int_{\mathbb{R}^3} \rho\, d\vec{x} = \frac{Q}{4\pi R^2} \cdot 4\pi \underbrace{\int_0^\infty r^2 \delta(r-R)\,dr}_{=R^2}$

$= Q$

b) $\cdot\ \rho(r,\varphi,z) = \frac{\lambda}{2\pi b}\,\delta(r-b)$

$\Rightarrow \frac{\text{Ladung}}{\text{Länge}} = \int_{\mathbb{R}^2} \rho\, d\vec{x} = \frac{\lambda}{2\pi b} \cdot 2\pi \int_0^\infty r\,\delta(r-b)\,dr = \lambda$