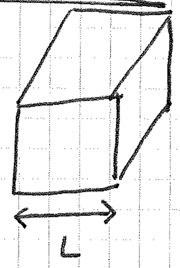
QM II - S4



$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2} \right)$$

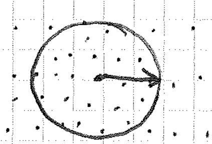
Periodische Randbedingungen $\qquad (n_x, n_y, n_z) \in \mathbb{Z}^3$

$$\sum_{i=1}^{N} \varepsilon_{\vec{k}}^{i} \; ; \quad \varepsilon_{\vec{k}}^{i} = \frac{\hbar^2 |\vec{k}|^2}{2m} \; ; \quad \vec{k} = \frac{2\pi}{L} \left(n_x \hat{e}_x + n_y \hat{e}_y + n_z \hat{e}_z \right)$$

a) ist einfach, kann weggelassen werden.

b)

$$n = \frac{N}{L^3} \qquad N = \sum_{|\vec{k}| \leq k_F} 2$$



Fermi-Kugel

Radius k_F

$$n = \frac{N}{L^3} = \sum_{|\vec{k}| < k_F} \left(\frac{2\pi}{L} \right)^3 \frac{2}{(2\pi)^3}$$

$$\xrightarrow{L \to \infty} \frac{2}{(2\pi)^3} \int_{|\vec{k}| < k_F} d^3k = \frac{2}{(2\pi)^3} \frac{4}{3} \pi k_F^3$$

$\#(\varepsilon)$ = Anzahl Zustände mit (Energie $< \varepsilon$)

$$= \frac{\text{Vol. d. Fermikugel mit } k_F \cdot 2}{\text{Vol. ersetzt von einem Zustand}} \longleftarrow \text{Spinentartung}$$

$\uparrow \left(\frac{2\pi}{L} \right)^3$

$\varepsilon + d\varepsilon$

$$\frac{d\#(\varepsilon)}{d\varepsilon} := D(\varepsilon)$$

$$D(\varepsilon)\, d\varepsilon = \#(\varepsilon + d\varepsilon) - \#(\varepsilon) \qquad (3) \Rightarrow (5) \text{ ableiten (!)}$$

$$N = \int_0^{\varepsilon_F} D(E)\, d\varepsilon$$

$N_\pm$ $\hat{=}$ Anzahl Teilchen mit Spin "↑" bzw. "Spin ↓"

$$N_+ = N_- = \frac{N}{2}$$

$$\parallel \int_0^{\varepsilon_F} D_+(\varepsilon)\, d\varepsilon = \frac{1}{2} \int_0^{\varepsilon_F} D(\varepsilon)\, d\varepsilon$$

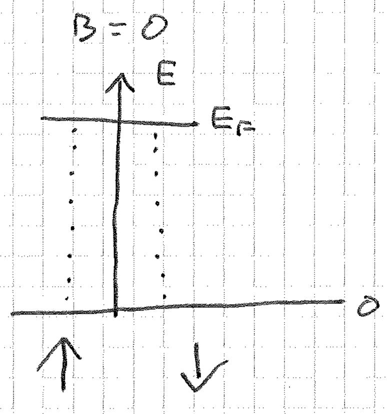
Nun: $H = H_{\text{freie } e^-} - \sum_{i=1}^{N} |\mu_B| g S_z^i B$ $\qquad \vec{B} = (0,0,B)^T$

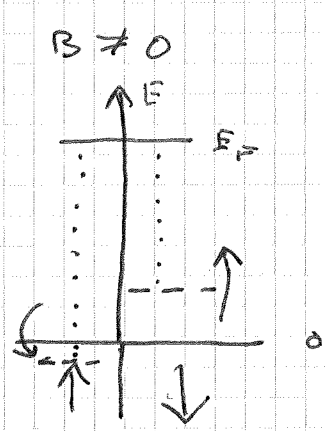
1-Teilchen-Zustand $|\varepsilon\uparrow\rangle \qquad |\varepsilon\downarrow\rangle$

$$H|\varepsilon\uparrow\rangle = (\varepsilon - |\mu_B| B)|\varepsilon\uparrow\rangle$$

$$H|\varepsilon\downarrow\rangle = (\varepsilon + |\mu_B| B)|\varepsilon\downarrow\rangle$$

$\boxed{T=0}$





$$\begin{cases} M = |\mu_B|(N_+ - N_-) \\ M = \underline{\underline{\chi}}\, B \end{cases}$$

$$N_+ = \int_{-|\mu_B| B}^{\varepsilon_F} D_+ (\varepsilon \pm |\mu_B| B)\, d\varepsilon$$

$$N_- = \int_{+|\mu_B| B}^{\varepsilon_F} D_- (\varepsilon \mp |\mu_B| B)\, d\varepsilon$$

Zusatzfrage: wieso trägt LB nicht bei?

Unetbootin
Foto links??

Starkeffekt

H^0 des Wasserstoffatoms

$\psi_{nlm} \propto R_n^l(r) Y_l^m(\vartheta, \varphi)$ Wellenfunktionen

$H = H^0 + V$ mit $V = -e|\vec{E}|z$

$-e|\vec{E}| \langle \psi_{nlm} | z | \psi_{nlm} \rangle$

Behauptung: z ist eine Komponente eines Tensoroperators vom Typ $(j = 1, \pi = -1)$

$$\left\{ T_{1;1}^{m} \right\}_{m = \{-1, 0, 1\}} \quad \text{mit} \quad T_{1\,-1}^{0} = z$$

Parität: $\mathbb{P}\varphi(\vec{x}) = \varphi(\mathbb{P}(\vec{x})) = \varphi(-\vec{x})$ $\quad \varphi \in L^2(\mathbb{R}^3)$

unitär

$$\mathbb{P}\,\psi_{nlm}(\vec{x}) = (-1)^l \psi_{nlm}(\vec{x})$$

$$\boxed{\text{weil } \mathbb{P} Y_l^m(\theta, \varphi) = Y_l^m(\pi + \theta, \pi - \varphi) = \ldots = (-1)^l Y_l^m(\theta, \varphi)}$$

$$\mathbb{P} z \mathbb{P}^* = -z$$

weil

$$\mathbb{P} z \mathbb{P} \varphi(\vec{x}) = \mathbb{P}(z \varphi(-\vec{x})) = -z\varphi(\vec{x})$$

$\mathbb{P} = \mathbb{P}^{-1} = \mathbb{P}^*$

$$\langle \psi_{nlm} | z | \psi_{n'l'm'} \rangle = \langle \psi_{nlm} | \underbrace{\mathbb{P}^* \mathbb{P}}_{1} z \underbrace{\mathbb{P}^* \mathbb{P}}_{1} | \psi_{n'l'm'} \rangle$$

$$= \langle \underbrace{\mathbb{P}\psi_{nlm}}_{(-1)^l \psi_{nlm}} | \underbrace{\mathbb{P} z \mathbb{P}^*}_{-z} | \underbrace{\mathbb{P}\psi_{n'l'm'}}_{(-1)^{l'} \psi_{n'l'm'}} \rangle$$

$$= (-1)^{l+l'+1} \langle \psi_{nlm} | z | \psi_{n'l'm'} \rangle = 0 \quad \text{falls } l + l' + 1 = \text{ungerade}$$

Wigner - Eckart

$T_{\tilde{\ell}}^{\tilde{m}}$ sei Tensoroperator vom Typ $(\tilde{\ell};\cdot)$

$$\langle \psi_{n\ell m} | T_{\tilde{\ell}}^{\tilde{m}} | \psi_{n'\ell'm'} \rangle = (\tilde{\ell}\,\tilde{m}\;\ell' m' | \ell m)\, \frac{(-1)^{2\tilde{\ell}}}{\sqrt{2\ell+1}}\, \underbrace{\langle n\ell \| T_{\tilde{\ell}} \| n'\ell' \rangle}_{\text{unabhängig von } m, m', \tilde{m}}$$

Man kann auch gehen, wenn's zu langweilig ist ...

... ich kann leider nicht gehen !!

Clebsch-Gordan-Koeffizienten (wann $\neq 0$??)

$$\mathcal{D}_{\tilde{\ell}} \otimes \mathcal{D}_{\ell'} = \bigoplus_{\ell=|\tilde{\ell}-\ell'|}^{\tilde{\ell}+\ell'} \mathcal{D}_{\ell}$$

$$\mathcal{D}_{\ell} \ni |\ell, m\rangle = \sum_{m'\tilde{m}} (\tilde{\ell}\,\tilde{m}\;\ell' m' | \ell m)\, \underset{\mathcal{D}_{\tilde{\ell}}}{\underset{\cap}{|\tilde{\ell}, \tilde{m}\rangle}} \otimes \underset{\mathcal{D}_{\ell'}}{\underset{\cap}{|\ell', m'\rangle}}$$

$= 0 =$ für

$$\ell \notin \{|\ell' - \tilde{\ell}|; \ldots; |\tilde{\ell}+\ell'|\}$$

$$\underset{\mathcal{D}_\ell}{L_3} = \underset{\mathcal{D}_{\tilde{\ell}}}{L_3} \otimes \mathbb{1} + \mathbb{1} \otimes \underset{\mathcal{D}_{\ell'}}{L_3}$$

$$L_3 |\ell, m\rangle = m |\ell, m\rangle$$

$$(L_3 \otimes \mathbb{1} + \mathbb{1} \otimes L_3)|\ell, m\rangle = \sum_{m'\tilde{m}} (\tilde{\ell}\,\tilde{m}\;\ell' m' | \ell m)\Big((\overbrace{L_3 |\tilde{\ell}\,\tilde{m}\rangle}^{\tilde{m}}) \otimes (\mathbb{1}|\ell' m'\rangle) + (\mathbb{1}|\tilde{\ell}, \tilde{m}\rangle) \otimes (\underbrace{L_3 |\ell' m'\rangle}_{m'})\Big)$$

$$= \sum_{m'\tilde{m}} (\tilde{m}+m')(\tilde{\ell}\,\tilde{m}\;\ell' m' | \ell m)\,(|\tilde{\ell}\,\tilde{m}\rangle \otimes |\ell' m'\rangle)$$

$$\overset{!}{=} m\,|\ell, m\rangle = m \sum_{m'\tilde{m}} (\ldots)\,(\ldots\ldots)$$

$$\Rightarrow \quad m = \tilde{m} + m'$$

$$CGK = 0 \quad \text{falls} \quad m \neq \tilde{m} + m' \quad !!$$

Stark-Effekt (back again ...)

$$\langle \psi_{nlm} | z | \psi_{n'l'm'} \rangle \propto \begin{pmatrix} 1 & 0 & l' & m' & l & m \end{pmatrix}$$

$\uparrow$ $T_{1\;-1}^{0}$ $\quad$ $\uparrow \hat{l}$ $\quad$ $\uparrow \hat{m}$

SU(2)-Symmetrie benutzt

$$\Rightarrow \quad m' = m, \qquad l = \begin{cases} l'-1 \\ l' \\ l'+1 \end{cases}$$

Parität killt noch $\cancel{l = l'}$

$$\Rightarrow \quad m' = m, \qquad l = \begin{cases} l'-1 \\ l'+1 \end{cases}$$

Warum $T_{1\;-1}^{0}$ Tensoroperator?

x, y, z $\quad$ $\left\{ T_{1\;-1}^{m} \right\}_{m = \{1, 0, -1\}}$

$$\left.\begin{aligned} T_{1\;-1}^{1} &= -\frac{(x+iy)}{\sqrt{2}} \\ T_{1\;-1}^{0} &= z \\ T_{1\;-1}^{-1} &= \frac{(x-iy)}{\sqrt{2}} \end{aligned}\right\} \text{Vektoroperatoren}$$

$$\vec{L} = \frac{1}{i}\, \vec{x} \wedge \vec{\nabla} \qquad L_3 \qquad L_\pm = L_1 \pm i L_2$$

Vorfaktoren „wegdefiniert"

$$[L_3, T_l^m] = m\, T_l^m$$

$$[L_\pm, T_l^m] = \sqrt{l(l+1) - m(m\pm 1)}\; T_l^{m\pm 1}$$

Beispielrechnung:

$$[L_3, T_1^1] = T_1^1 \qquad m = 1 \qquad L_3 = \frac{1}{i}(x\partial_y - y\partial_x)$$

$$[L_3, T_1^1]\, \psi(\vec{x}) = \frac{-1}{\sqrt{2}\, i} (y\partial_x - x\partial_y)(x+iy)\, \psi(\vec{x})$$

$$- \frac{1}{\sqrt{2}\, i} (x+iy)(y\partial_x - x\partial_y)\, \psi(\vec{x})$$

$$= \frac{-1}{\sqrt{2}\, i} \Big[(y - ix)\, \psi(\vec{x}) + (x+iy)(y\partial_x - x\partial_y)\, \psi(\vec{x})$$

$$+ \text{2. Term} \Big]$$

$$= \ldots =$$

$$= T_1^1\, \psi(x)$$

→ Kugelkoordinaten

$$T^{1}_{1\,-1} = \frac{-1r}{\sqrt{2}} \sin\theta (\cos\varphi + i \sin\varphi) \propto r\, Y^1_1(\theta,\varphi)$$
$$T^{0}_{1\,-1} = z = r\cos\theta \qquad \propto r\, Y^1_0(\theta,\varphi)$$
$$T^{1}_{1\,-1} = \frac{1}{\sqrt{2}}\, r \sin\theta(-\cos\varphi - i\sin\varphi) \propto r\, Y^1_{-1}(\theta,\varphi)$$

proportional zu Kugelfunktionen, daher ist die Wahl für T_ℓ^m natürlich.

→ Auch für höhere m und ℓ möglich!

$\vec{L}$ definiert auch Tensoroperator:

$$T^{1}_{1,1} = -\frac{1}{\sqrt{2}}(L_1 + i L_2) = -\frac{1}{\sqrt{2}} L_+$$
$$T^{0}_{1,1} = L_3$$
$$T^{-1}_{1,1} = \frac{1}{\sqrt{2}}(L_1 - i L_2) = \frac{1}{\sqrt{2}} L_-$$