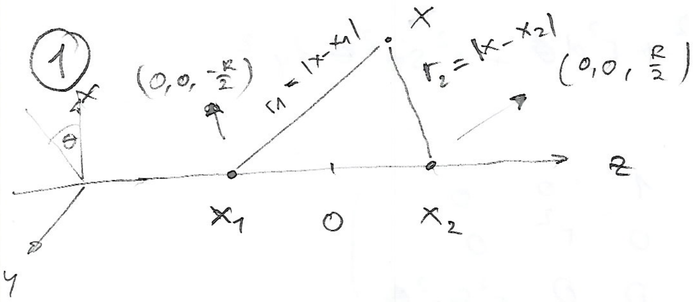
QM Tipps 8

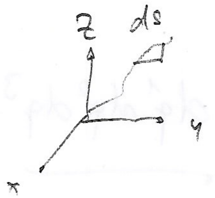
(1)



$$H_{el}(x_1, x_2) = -\frac{\hbar^2 \Delta x}{2m} - \frac{e^2}{4\pi |x_1 - x|} - \frac{e^2}{4\pi |x_2 - x|}$$

$$H_{el}\, \varphi(x) = E \varphi(x)$$

Koord Transf.



$$(x, y, z) = (x^1, x^2, x^3)$$

$$(\lambda, \mu, \varphi) = (q^1, q^2, q^3)$$

$$x^i = f^i(q^1, q^2, q^3), \ \forall i$$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $\exists f^{-1}$

f Diffeomorphismus

$$ds^2 = dx^2 + dy^2 + dz^2 = g_{ij}\, dq^i dq^j$$

g_{ij} : Metrik, symmetrisch

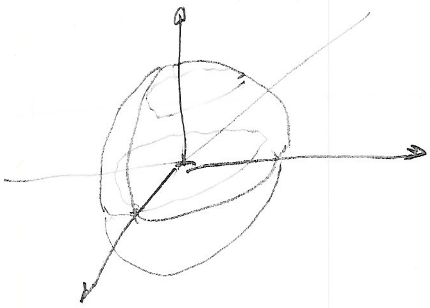
$\det g_{ij} \neq 0$

$$dx^1 = \frac{\partial x^1}{\partial q^1} dq^1 + \frac{\partial x^1}{\partial q^2} dq^2 \ldots$$

$$dx^i = \sum_{k=1}^{3} \frac{\partial x^i}{\partial q^k} dq^k, \ \ldots \Rightarrow g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k \partial x^k}{\partial q^i \partial q^j}$$

Jetzt: triviales Bsp. mit Kugelkoordinaten:

Wir müssen dann genau die gleichen Schritte ausführen, aber mit den Koordinaten der Aufgabe



Orthogonale Koord. $\Leftrightarrow$ Metrik diagonal

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$\Rightarrow g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

<u>Integral</u> $U \subseteq \mathbb{R}^3, \quad q^i, \quad g$

$$h : U \to \mathbb{R}$$

$$\int_U h \, dV := \int_U \underbrace{\sqrt{|g|} \, dq^1 dq^2 dq^3}\, h(q^1, q^2, q^3)$$

mit $|g| = |\det g_{ij}|$ Riemann'sches Volumen Element

<u>Laplace</u>

(*) $$\int f (\Delta g) \, dV = - \int \nabla f \, \nabla g \, dV \Rightarrow \Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial q^i} \left(\sqrt{|g|} \, g^{ij} \frac{\partial}{\partial q^j} \right)$$

$$\nabla_x = \left(\frac{\partial}{\partial x^1}, \partial_{x^2}, \partial_{x^3} \right)$$

$$\nabla f \cdot \nabla g = \sum_{k=1}^{3} \frac{\partial f}{\partial x^k} \cdot \frac{\partial g}{\partial x^k} = \sum_{k=1}^{3} \frac{\partial q^l}{\partial x^k} \frac{\partial f}{\partial q^l} \cdot \frac{\partial q^m}{\partial q^k} \frac{\partial g}{\partial q^m}$$

$$\left[\frac{\partial}{\partial x^k} = \frac{\partial q^l}{\partial x^k} \frac{\partial}{\partial q^l} \right]$$

$$= \underbrace{\left(\sum_{k=1}^{3} \frac{\partial q^l}{\partial x^k} \frac{\partial q^m}{\partial x^k} \right)}_{g^{lm}} \frac{\partial f}{\partial q^l} \cdot \frac{\partial g}{\partial q^m}$$

R.S. von (*) in Koord q^l

Partielle Integration

$$-\int \sqrt{|g|}\, dq^1 dq^2 dq^3 \frac{\partial f}{\partial q^l} g^{lm} \frac{\partial g}{\partial q^m}$$

$$= \int dq^1 dq^2 dq^3 \, f \cdot \frac{\partial}{\partial q^l} \left(\sqrt{|g|}\, g^{lm} \frac{\partial}{\partial q^m} \right) g$$

$\uparrow \frac{\sqrt{|g|}}{\sqrt{|g|}}$

$$= \int dV f \underbrace{\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial q^l} \left(\sqrt{|g|}\, g^{lm} \frac{\partial}{\partial q^m} \right)}\, g$$

also ist das der Δ in bel. Koordinaten

Kugelkoordinaten:

$$\Delta = \frac{1}{r^2 \sin\vartheta} \frac{\partial}{\partial r} \left(r^2 \sin\vartheta \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\vartheta} \frac{\partial}{\partial \vartheta} \left(r^2 \sin\vartheta \frac{1}{r^2} \frac{\partial}{\partial \vartheta} \right)$$

$$+ \frac{1}{r^2 \sin\vartheta} \frac{\partial}{\partial \varphi} \left(r^2 \sin\vartheta \frac{1}{r^2 \sin^2\vartheta} \frac{\partial}{\partial \varphi} \right)$$

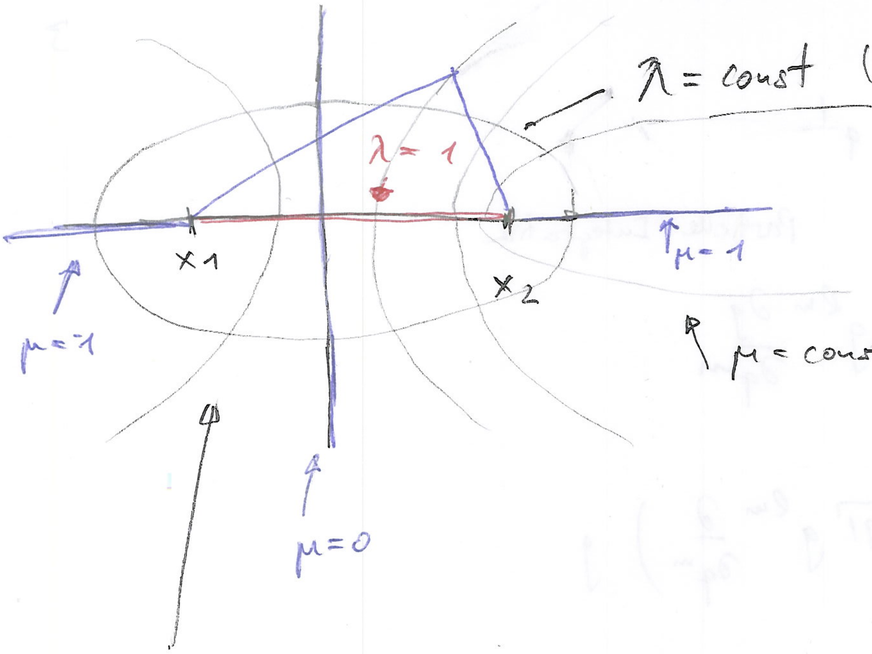
in Aufgabe 1: Do the same mit anderen Koordinaten:

$\lambda := \frac{r_1 + r_2}{R}$ | $x = \ldots \lambda, \mu, \phi$

$\mu := \frac{r_1 - r_2}{R}$ | $y =$

$\phi =$ Winkel siehe Zeichnung | $z =$

↑ Rechne!

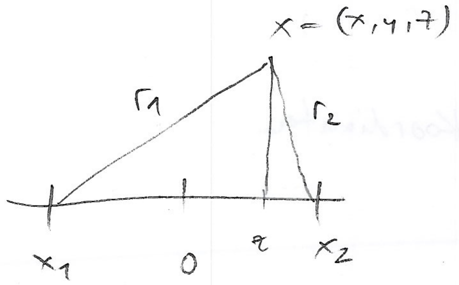


$\lambda \in [1, \infty)$

$\mu \in [-1, 1]$

$\phi \in [0, 2\pi)$

Kurven schneiden sich rechtwinklig ⇒ Metrik wird diagonal sein



$x = \cos\phi \, r$

$y = \sin\phi \, r$

$r^2 = x^2 + y^2$

$r_1^2 = r^2 +$

$r_2^2 = r^2 + \left(\frac{R}{2} - z\right)^2$

... Löse jetzt 8.1.i-iii

iv) Setze den Δ elliptisch ein im den Hamiltonian und verwende Separation der Variablen

Bsp Separation der Variablen (Kugel koord)

$$\Delta\phi = 0 \qquad \Phi = \frac{U(r)}{r} P(\theta)\, Q(\varphi)$$

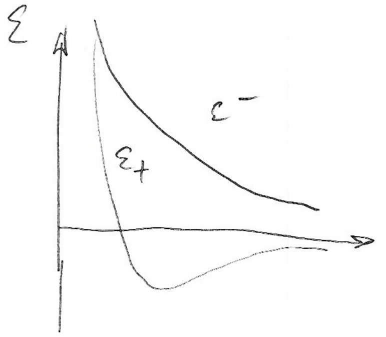
$$= \frac{PQ}{r}\frac{d^2U}{dr^2} + \frac{UQ}{r\cdot r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial P}{\partial\theta} + \frac{1}{r}UP\frac{1}{r^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0 \quad \Big| \, r^2\sin^2\theta$$

. . . .

(3)

$$\left(\frac{-\hbar^2\Delta_{x_1}}{2m_1} + \frac{-\hbar^2\Delta_{x_2}}{2m_2} + \varepsilon(R) \right)\phi(x_1,x_2) = E\,\phi(x_1,x_2)$$

$\varepsilon(R) = e(R) + V_{KK}, \quad R = x_1 - x_2$



Relativbewegung herausnehmen

$$\leadsto -\frac{\hbar\Delta_R}{2\mu} + \varepsilon_+(|R|)$$

$$\Delta_R = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{L^2}{r^2}$$

$$\leadsto \frac{-\hbar^2}{\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right)$$

$$\frac{u_{en}}{r} Y_l^m(\theta,\varphi)$$

$$V_{eff} = \varepsilon(|R|) + \frac{\hbar^2 l(l+1)}{2\mu R^2}$$