1 Koordinatensysteme

1.1 Koordinaten

Kartesisch	Sphärisch	Zylindrisch
$x = r \cdot \sin(\theta) \cdot \cos(\phi)$ $y = r \cdot \sin(\theta) \cdot \sin(\phi)$ $z = r \cdot \cos(\theta)$	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arccos(\frac{z}{r})$ $\phi = \arctan(\frac{y}{x})$	$\rho = \sqrt{x^2 + y^2}$ $\phi = \arctan(\frac{y}{x})$ $z = z$
$x = \rho \cdot \cos(\phi)$ $y = \rho \cdot \sin(\phi)$ $z = z$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan(\frac{\rho}{z})$ $\phi = \phi$	$\rho = r \cdot \sin(\theta)$ $\phi = \phi$ $z = r \cdot \cos(\theta)$

1.2 Einheitsvektoren

Kartesisch	Sphärisch	Zylindrisch
$\mathbf{n}_x = \sin(\theta) \cdot \cos(\phi) \cdot \mathbf{n}_r +$ $\cos(\theta) \cdot \cos(\phi) \cdot \mathbf{n}_\theta -$ $\sin(\phi) \cdot \mathbf{n}_\phi$ $\mathbf{n}_y = \sin(\theta) \cdot \sin(\phi) \cdot \mathbf{n}_r +$ $\cos(\theta) \cdot \sin(\phi) \cdot \mathbf{n}_\theta +$ $\cos(\phi) \cdot \mathbf{n}_\phi$ $\mathbf{n}_z = \cos(\theta) \cdot \mathbf{n}_r -$ $\sin(\theta) \cdot \mathbf{n}_\theta$	$\mathbf{n}_r = \frac{x}{r} \cdot \mathbf{n}_x +$ $\frac{y}{r} \cdot \mathbf{n}_y +$ $\frac{z}{r} \cdot \mathbf{n}_z$ $\mathbf{n}_\theta = \frac{x \cdot z}{r \cdot \sqrt{x^2+y^2}} \cdot \mathbf{n}_x +$ $\frac{y \cdot z}{r \cdot \sqrt{x^2+y^2}} \cdot \mathbf{n}_y -$ $\frac{x^2+y^2}{r \cdot \sqrt{x^2+y^2}} \cdot \mathbf{n}_z$ $\mathbf{n}_\phi = \frac{-y}{\sqrt{x^2+y^2}} \cdot \mathbf{n}_x +$ $\frac{x}{\sqrt{x^2+y^2}} \cdot \mathbf{n}_y$	$\mathbf{n}_\rho = \frac{x}{\rho} \cdot \mathbf{n}_x +$ $\frac{y}{\rho} \cdot \mathbf{n}_y$ $\mathbf{n}_\phi = \frac{-y}{\rho} \cdot \mathbf{n}_x +$ $\frac{x}{\rho} \cdot \mathbf{n}_y$ $\mathbf{n}_z = \mathbf{n}_z$
$\mathbf{n}_x = \cos(\phi) \cdot \mathbf{n}_\rho -$ $\sin(\phi) \cdot \mathbf{n}_\phi$ $\mathbf{n}_y = \sin(\phi) \cdot \mathbf{n}_\rho +$ $\cos(\phi) \cdot \mathbf{n}_\phi$ $\mathbf{n}_z = \mathbf{n}_z$	$\mathbf{n}_r = \frac{\rho}{r} \cdot \mathbf{n}_\rho +$ $\frac{z}{r} \cdot \mathbf{n}_z$ $\mathbf{n}_\theta = \frac{z}{r} \cdot \mathbf{n}_\rho -$ $\frac{\rho}{r} \cdot \mathbf{n}_z$ $\mathbf{n}_\phi = \mathbf{n}_\phi$	$\mathbf{n}_\rho = \sin(\theta) \cdot \mathbf{n}_r +$ $\cos(\theta) \cdot \mathbf{n}_\theta$ $\mathbf{n}_\phi = \mathbf{n}_\phi$ $\mathbf{n}_z = \cos(\theta) \cdot \mathbf{n}_r -$ $\sin(\theta) \cdot \mathbf{n}_\theta$

1.3 Orthogonalitätsrelationen

Kartesisch	Sphärisch	Zylindrisch
$\mathbf{n}_x = \mathbf{n}_y \times \mathbf{n}_z$	$\mathbf{n}_r = \mathbf{n}_\theta \times \mathbf{n}_\phi$	$\mathbf{n}_\rho = \mathbf{n}_\phi \times \mathbf{n}_z$
$\mathbf{n}_y = \mathbf{n}_z \times \mathbf{n}_x$	$\mathbf{n}_\theta = \mathbf{n}_\phi \times \mathbf{n}_r$	$\mathbf{n}_\phi = \mathbf{n}_z \times \mathbf{n}_\rho$
$\mathbf{n}_z = \mathbf{n}_x \times \mathbf{n}_y$	$\mathbf{n}_\phi = \mathbf{n}_r \times \mathbf{n}_\theta$	$\mathbf{n}_z = \mathbf{n}_\rho \times \mathbf{n}_\phi$

2 Differentialoperatoren

2.1 Gradient ∇f

Kartesisch	$\nabla f = \partial_x f \cdot \mathbf{n}_x + \partial_y f \cdot \mathbf{n}_y + \partial_z f \cdot \mathbf{n}_z$
Sphärisch	$\nabla f = \partial_r f \cdot \mathbf{n}_r + \frac{1}{r} \cdot \partial_\theta f \cdot \mathbf{n}_\theta + \frac{1}{r \sin\theta} \cdot \partial_\phi f \cdot \mathbf{n}_\phi$
Zylindrisch	$\nabla f = \partial_\rho f \cdot \mathbf{n}_\rho + \frac{1}{\rho} \cdot \partial_\phi f \cdot \mathbf{n}_\phi + \partial_z f \cdot \mathbf{n}_z$

2.2 Divergenz $\nabla \cdot \mathbf{A}$

Kartesisch	$\nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z$
Sphärisch	$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \cdot \partial_r \left(r^2 \cdot A_r\right) + \frac{1}{r \sin\theta} \cdot \partial_\theta (\sin\theta \cdot A_\theta) + \frac{1}{r \sin\theta} \cdot \partial_\phi A_\phi$
Zylindrisch	$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \cdot \partial_\rho (\rho \cdot A_\rho) + \frac{1}{\rho} \cdot \partial_\phi A_\phi + \partial_z A_z$

2.3 Rotation $\nabla \times \mathbf{A}$

Kartesisch	$\nabla \times \mathbf{A} = (\partial_y A_z - \partial_z A_y) \cdot \mathbf{n}_x + (\partial_z A_x - \partial_x A_z) \cdot \mathbf{n}_y +$ $(\partial_x A_y - \partial_y A_x) \cdot \mathbf{n}_z$
Sphärisch	$\nabla \times \mathbf{A} = \frac{1}{r \sin\theta} \cdot (\partial_\theta (\sin\theta \cdot A_\phi) - \partial_\phi A_\theta) \cdot \mathbf{n}_r +$ $\frac{1}{r} \cdot \left(\frac{1}{\sin\theta} \partial_\phi A_r - \partial_r (r \cdot A_\phi)\right) \cdot \mathbf{n}_\theta +$ $\frac{1}{r} \cdot (\partial_r (r \cdot A_\theta) - \partial_\theta A_r) \cdot \mathbf{n}_\phi$
Zylindrisch	$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \partial_\phi A_z - \partial_z A_\phi\right) \cdot \mathbf{n}_\rho + (\partial_z A_\rho - \partial_\rho A_z) \cdot \mathbf{n}_\phi +$ $\frac{1}{\rho} \cdot (\partial_\rho (\rho \cdot A_\phi) - \partial_\phi A_\rho) \cdot \mathbf{n}_z$

2.4 Skalarer Laplace $\nabla^2 f$

Kartesisch	$\nabla^2 f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$
Sphärisch	$\nabla^2 f = \frac{1}{r^2} \cdot \partial_r \left(r^2 \cdot \partial_r f\right) + \frac{1}{r^2 \sin\theta} \cdot \partial_\theta (\sin\theta \cdot \partial_\theta f) +$ $\frac{1}{r^2 \sin^2\theta} \cdot \partial_\phi^2 f$
Zylindrisch	$\nabla^2 f = \frac{1}{\rho} \cdot \partial_\rho (\rho \cdot \partial_\rho f) + \frac{1}{\rho^2} \cdot \partial_\phi^2 f + \partial_z^2 f$

2.5 Vektorieller Laplace $\nabla^2\mathbf{A}$

Kartesisch	$\nabla^2\mathbf{A} = \nabla^2 A_x \cdot \mathbf{n}_x + \nabla^2 A_y \cdot \mathbf{n}_y + \nabla^2 A_z \cdot \mathbf{n}_z$
Sphärisch	$\nabla^2\mathbf{A} = \big(\nabla^2 A_r - \frac{2}{r^2} \cdot A_r - \frac{2}{r^2 \sin\theta} \cdot \partial_\theta \left(\sin\theta \cdot A_\theta\right) - \frac{2}{r^2 \sin\theta} \cdot \partial_\phi A_\phi\big) \cdot \mathbf{n}_r + \big(\nabla^2 A_\theta - \frac{A_\theta}{r^2 \sin^2\theta} + \frac{2}{r^2} \cdot \partial_\theta A_r - \frac{2\cos\theta}{r^2 \sin^2\theta} \cdot \partial_\phi A_\phi\big) \cdot \mathbf{n}_\theta + \Big(\nabla^2 A_\phi - \frac{A_\phi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta} \cdot \partial_\phi A_r + \frac{2\cos\theta}{r^2 \sin^2\theta} \cdot \partial_\phi A_\theta\Big) \cdot \mathbf{n}_\phi$
Zylindrisch	$\nabla^2\mathbf{A} = \Big(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2} \cdot \partial_\phi A_\phi\Big) \cdot \mathbf{n}_\rho + \Big(\nabla^2 A_\phi - \frac{A_\phi}{\rho^2} - \frac{2}{\rho^2} \cdot \partial_\phi A_\rho\Big) \cdot \mathbf{n}_\phi + \nabla^2 A_z \cdot \mathbf{n}_z$

2.6 Identitäten

$$\begin{aligned}
\nabla(f \cdot g) &= f \cdot \nabla g + g \cdot \nabla f \\
\nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \\
\nabla \cdot (f \cdot \mathbf{A}) &= f \cdot (\nabla \cdot A) + \mathbf{A} \cdot \nabla f \\
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\
\nabla \cdot (\nabla \times \mathbf{A}) &= 0 \\
\nabla \times (f \cdot \mathbf{A}) &= f \cdot (\nabla \times \mathbf{A}) + \nabla f \times \mathbf{A} \\
\nabla \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A} \cdot (\nabla \cdot \mathbf{B}) - \mathbf{B} \cdot (\nabla \cdot \mathbf{A}) \\
\nabla \times \nabla \times \mathbf{A} &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \\
\nabla \times \nabla f &= \mathbf{0}
\end{aligned}$$

3 Integraloperationen

3.1 Differentielle Elemente

$$
\begin{aligned}
\mathrm{d}\mathbf{s} &= \mathrm{d}x \cdot \mathbf{n}_x + \mathrm{d}y \cdot \mathbf{n}_y + \mathrm{d}z \cdot \mathbf{n}_z \\
&= \mathrm{d}r \cdot \mathbf{n}_r + r \cdot \mathrm{d}\theta \cdot \mathbf{n}_\theta + r \sin\theta \cdot \mathrm{d}\phi \cdot \mathbf{n}_\phi \\
&= \mathrm{d}\rho \cdot \mathbf{n}_\rho + \rho \cdot \mathrm{d}\phi \cdot \mathbf{n}_\phi + \mathrm{d}z \cdot \mathbf{n}_z \\
\mathrm{d}\mathbf{a} &= \mathrm{d}y \cdot \mathrm{d}z \cdot \mathbf{n}_x + \mathrm{d}x \cdot \mathrm{d}z \cdot \mathbf{n}_y + \mathrm{d}x \cdot \mathrm{d}y \cdot \mathbf{n}_z \\
&= r^2 \sin\theta \cdot \mathrm{d}\theta \cdot \mathrm{d}\phi \cdot \mathbf{n}_r + r \sin\theta \cdot \mathrm{d}r \cdot \mathrm{d}\phi \cdot \mathbf{n}_\theta + r \cdot \mathrm{d}r \cdot \mathrm{d}\theta \cdot \mathbf{n}_\phi \\
&= \rho \cdot \mathrm{d}\phi \cdot \mathrm{d}z \cdot \mathbf{n}_\rho + \mathrm{d}\rho \cdot \mathrm{d}z \cdot \mathbf{n}_\phi + \rho \cdot \mathrm{d}\rho \cdot \mathrm{d}\phi \cdot \mathbf{n}_z \\
\mathrm{d}V &= \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z \\
&= r^2 \sin\theta \cdot \mathrm{d}r \cdot \mathrm{d}\theta \cdot \mathrm{d}\phi \\
&= \rho \cdot \mathrm{d}\rho \cdot \mathrm{d}\phi \cdot \mathrm{d}z
\end{aligned}
$$

3.2 Kurvenintegrale

Skalar (1. Art): $\int\limits_\gamma f(\mathbf{x})\,\mathrm{d}s = \int\limits_a^b f(\gamma(t))\,\|\dot{\gamma}(t)\|_2\ \mathrm{d}t$

Vektoriell (2. Art): $\int\limits_\gamma \mathbf{f}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \int\limits_a^b \mathbf{f}(\gamma(t)) \cdot \dot{\gamma}(t)\,\mathrm{d}t$

3.3 Oberflächenintegrale

Skalar: $\iint\limits_S f(\mathbf{x})\,\mathrm{d}a = \iint\limits_B f\left(\varphi(u,v)\right) \cdot ||\partial_u\varphi \times \partial_v\varphi||\ \mathrm{d}\mu(u,v)$

Vektoriell: $\iint\limits_S \mathbf{f}(\mathbf{x})\,\mathrm{d}\mathbf{a} = \iint\limits_B f\left(\varphi(u,v)\right) \cdot \left(\partial_u\varphi \times \partial_v\varphi\right)\ \mathrm{d}\mu(u,v)$

3.4 Volumenintegrale

$$\iiint\limits_V f(\mathbf{x})\,\mathrm{d}V = \iiint\limits_B f\left(\xi(u,v,w)\right) \cdot \left|\det\left(J_\xi(u,v,w)\right)\right|\ \mathrm{d}\mu(u,v,w)$$

$$J_\xi(u,v,w) = \begin{pmatrix}\partial_u\xi & \partial_v\xi & \partial_w\xi\end{pmatrix} = \begin{pmatrix}\partial_u\xi_x & \partial_v\xi_x & \partial_w\xi_x \\ \partial_u\xi_y & \partial_v\xi_y & \partial_w\xi_y \\ \partial_u\xi_z & \partial_v\xi_z & \partial_w\xi_z\end{pmatrix}$$

3.5 Satz von Gauss

$$\oiint\limits_{\partial V} \mathbf{A}\,\mathrm{d}\mathbf{a} = \iiint\limits_V \nabla \cdot \mathbf{A}\,\mathrm{d}V$$

3.6 Satz von Stokes

$$\oint\limits_{\partial S} \mathbf{A}\,\mathrm{d}\mathbf{s} = \iint\limits_S \nabla \times \mathbf{A}\,\mathrm{d}\mathbf{a}$$